Chapters 4.2 Drawing graphs in the plane

Informally, drawing of a graph G in the plane is assignment of distinct points to the vertices and curves to edges such that curves have as endpoints their vertices and curves intersect only at endpoints. A very formal definition is in the book, please read it!

A graph G is **planar** if it is possible to draw it in the plane (without crossings of edges - except their endpoints).

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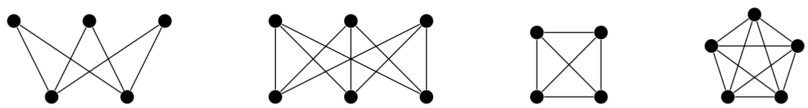
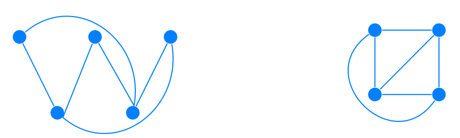
1: Are the following graphs $K_{2,3}$, $K_{3,3}$, K_4, and K_5 planar graphs?

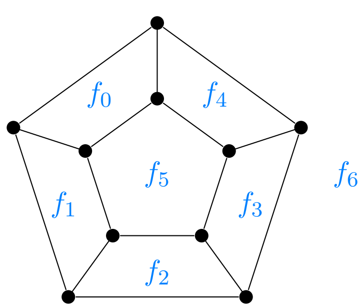


Solution: $K_{2,3}$ yes $K_{3,3}$ no K_4 yes K_5 no



A **face** (sometimes called **region**) in a plane graph G is a region of the plane that is obtained by removing the edges (and vertices) of G from the plane. (Imagine drawing G on a paper and cutting along the edges. The connected pieces of the paper after the cuttings is done are called faces.)

2: Mark individual faces in the following plane graph. How many faces does it have?



The unbounded piece is called **outer/exterior face/region**.

Drawings can be very wild, but there is always a *simple* one.

Theorem If G is a planar graph, then it has a drawing where all edges correspond to straight line segments.

3: Draw $K_{2,2}$ and K_4 using only straight lines.

Solution: One has to move vertices.

Theorem - Euler's Identity Let G be a *connected* plane graph with $v \geq 1$ vertices, e edges and f faces. Then

$$v + f = e + 2.$$

4: Verify that Euler's identity holds for $K_{2,3}$, K_4 and all trees. Notice that trees are planar graphs.

Solution: We will use drawings of K_4 and $K_{2,3}$ that we already have to count the faces. Denote the set of faces of G by $F(G)$.
$|V(K_4)| = 4$, $|F(K_4)| = 4$, $|E(K_4)| = 6$
$|V(K_{2,3})| = 5$, $|F(K_{2,3})| = 3$, $|E(K_{2,3})| = 6$
Recall that for every tree T, $|E(T) = |V(T)| - 1$ and notice that in any drawing $|F(T)| = 1$.

5: Prove Euler's Identity. Use induction and that every graph can be created from one vertex by adding leaves and edges.

Solution: If G has one vertex, zero edges and one face, the identity holds. If G has a cycle C, then removing one edge from the cycle decreases the number of edges by one and number of faces by one. If G has a vertex of degree one, then removing the vertex and its incident edge decreases the number of edges by one and number of vertices by one. Notice that both cases change both sides of the equation by one.

6: Let G be a plane graph with f faces and e edges, where $e \geq 2$. Show that $3f \leq 2e$.
Hint: Counting (edge side)-face incidences should do it.

Solution: Let x be the number of (edge side)-face incidencies. This way, every edge is incident with two faces (or one face twice if it is a bridge) and we get $2e = x$. On the other hand, the smallest face is a triangle, hence $3f \leq x$. This gives $3f \leq 2e$.

Theorem. If G is a planar graph of order at least 3, then

$$|E(G)| \leq 3|V(G)| - 6.$$

7: Prove Theorem.
Hint: Use Euler and get rid of f using previous question.

Solution: If $|E(G)| \leq 3$, the inequality holds. We use the Euler Identity $v+f = e+2$ and combine it with $3f \leq 2e$. That gives $3v + 3f = 3e + 6$ and $2e + 3v \geq 3e + 6$, which is the same as $e \leq 3v - 6$. What happens if G is not connected?

8: Show that K_5 is not a planar graph.
Hint: Use Euler's Formula

Solution: K_5 has 10 edges and 5 vertices. So it does not satisfy that $10 \leq 3 \cdot 5 - 6$.

9: Show that every planar graph has a vertex of degree at most 5.
Hint: Use Eulers formula and the handshaking lemma.

Solution: Suppose for contradiction that G has a minimum degree 6. Then $2|E(G)| = \sum_{v \in V(G)} \deg v \geq 6|V(G)|$, which contradicts that $|E(G)| \leq 3|V(G)| - 6$.

10: Show that if G is a bipartite planar graph with at least one two edges then

$$|E(G)| \leq 2|V(G)| - 4.$$

Solution: Let G be a bipartite plane graph with f faces, v vertices and e edges,

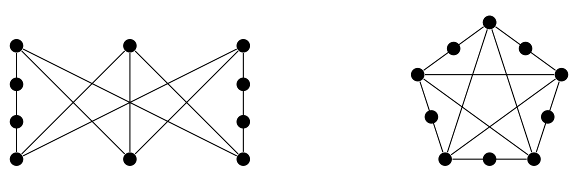
If G is bipartite and not trivial, then the smallest face has at least 4 edges. Hence $4f \leq 2e$.

We use the Euler Identity $v + f = e + 2$ and combine it with $4f \leq 2e$. That gives $2v + 2f = 2e + 4$ and $e + 2v \geq 2e + 4$, which is the same as $e \leq 2v - 4$.

11: Show that $K_{3,3}$ is not a planar graph.

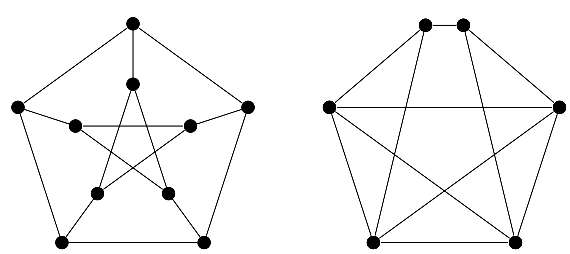
Solution: $K_{3,3}$ has 9 edges and 6 vertices. Hence it does not satisfy $e \leq 2v - 4$.

12: Are the following graphs planar? Why?



Solution: No. If they were planar, one could draw K_5 or $K_{3,3}$ as a planar graph.

13: Is Petersen's graph planar? And the one next to it?



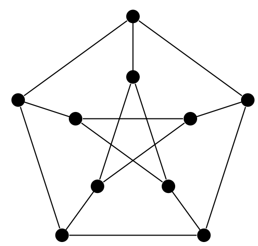
Solution: Petersen contains a subdivision of $K_{3,3}$. The other one looks basically like K_5. See the minor explanation below.

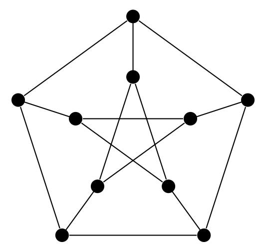
Theorem - Kuratowski A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Let G be a graph. A graph H is a **minor** of G if H can be obtained from G by deleting vertices, deleting edges and contracting edges.

Theorem A graph G is planar iff it does not contain K_5 or $K_{3,3}$ as a minor.

14: Show that Petersen's graph has K_5 as a minor and also $K_{3,3}$ as a minor.





15: Are the following graphs planar?